Special Instructions / Useful Data

$\mathbb{Z}_n = \{\bar{0}, \bar{1}, \dots, \overline{n-1}\}$ denotes the additive group of integers modulo n

$\mathbb{R} =$ the set of all real numbers

$\mathbb{N} =$ the set of all positive integers

$\mathbb{Z} =$ the set of all integers

$\mathbb{C} =$ the set of all complex numbers

$\mathbb{Q} =$ the set of all rational numbers

$\gcd(r, n) =$ the greatest common divisor of the integers r and n

$S_n =$ the symmetric group of all permutations of $\{1, 2, \dots, n\}$

A_n = the group of all even permutations in S_n

$M_n(\mathbb{C}) =$ the set of all $n \times n$ matrices with entries from $\mathbb{C}$

$M_n(\mathbb{R}) =$ the set of all $n \times n$ matrices with entries from $\mathbb{R}$

$M^T =$ the transpose of the matrix M

$I_n =$ the $n \times n$ identity matrix

$P_n(x) =$ the real vector space of polynomials, in the variable x with real coefficients and having degree at most n, together with the zero polynomial. These polynomials are regarded as functions from $\mathbb{R}$ to $\mathbb{R}$

$\binom{n}{k} =$ the binomial coefficient defined as $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$f \circ g =$ the composite function defined by $(f \circ g)(x) = f(g(x))$

$A \setminus B$ = the complement of the set B in the set A, that is, $\{x \in A : x \notin B\}$

$\log x =$ the logarithm of x to the base e for a positive number x

$\mathbb{R}^n =$ the n-dimensional Euclidean space

$A \times B$ = the Cartesian product of the sets A and B

$M^{-1} =$ the inverse of an invertible matrix M

Section A: Q.1 – Q.10 Carry ONE mark each.

Q.1 Let $y_c: \mathbb{R} \to (0, \infty)$ be the solution of the Bernoulli's equation

$$\frac{dy}{dx} - y + y^3 = 0, \qquad y(0) = c > 0.$$

Then, for every $c > 0$, which one of the following is true?

(A) $\lim_{x\to\infty} y_c(x) = 0$

(B) $\lim_{x\to\infty} y_c(x) = 1$

(C) $\lim_{x\to\infty} y_c(x) = e$

(D) $\lim_{x\to\infty} y_c(x)$ does not exist

Q.2 For a twice continuously differentiable function $g: \mathbb{R} \to \mathbb{R}$, define

$$u_g(x, y) = \frac{1}{y} \int_{-y}^{y} g(x+t)dt \quad \text{for } (x, y) \in \mathbb{R}^2, \qquad y > 0.$$

Which one of the following holds for all such g?

(A) $\dfrac{\partial^2 u_g}{\partial x^2} = \dfrac{2}{y} \dfrac{\partial u_g}{\partial y} + \dfrac{\partial^2 u_g}{\partial y^2}$

(B) $\dfrac{\partial^2 u_g}{\partial x^2} = \dfrac{1}{y} \dfrac{\partial u_g}{\partial y} + \dfrac{\partial^2 u_g}{\partial y^2}$

(C) $\dfrac{\partial^2 u_g}{\partial x^2} = \dfrac{2}{y} \dfrac{\partial u_g}{\partial y} - \dfrac{\partial^2 u_g}{\partial y^2}$

(D) $\dfrac{\partial^2 u_g}{\partial x^2} = \dfrac{1}{y} \dfrac{\partial u_g}{\partial y} - \dfrac{\partial^2 u_g}{\partial y^2}$

Q.3 Let $y(x)$ be the solution of the differential equation

$$\frac{dy}{dx} = 1 + y \sec x \quad \text{for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

that satisfies $y(0) = 0$. Then, the value of $y\left(\frac{\pi}{6}\right)$ equals

(A) $\sqrt{3}\ \log\left(\frac{3}{2}\right)$

(B) $\left(\frac{\sqrt{3}}{2}\right) \log\left(\frac{3}{2}\right)$

(C) $\left(\frac{\sqrt{3}}{2}\right) \log 3$

(D) $\sqrt{3}\ \log 3$

Q.4 Let $\mathcal{F}$ be the family of curves given by

$$x^2 + 2hxy + y^2 = 1, \quad -1 < h < 1\,.$$

Then, the differential equation for the family of orthogonal trajectories to $\mathcal{F}$ is

(A) $(x^2y - y^3 + y)\dfrac{dy}{dx} - (xy^2 - x^3 + x) = 0$

(B) $(x^2y - y^3 + y)\dfrac{dy}{dx} + (xy^2 - x^3 + x) = 0$

(C) $(x^2y + y^3 + y)\dfrac{dy}{dx} - (xy^2 + x^3 + x) = 0$

(D) $(x^2y + y^3 + y)\dfrac{dy}{dx} + (xy^2 + x^3 + x) = 0$

Q.5 Let G be a group of order 39 such that it has exactly one subgroup of order 3 and exactly one subgroup of order 13. Then, which one of the following statements is TRUE?

(A) G is necessarily cyclic

(B) G is abelian but need not be cyclic

(C) G need not be abelian

(D) G has 13 elements of order 13

Q.6 For a positive integer n, let $U(n) = \{\bar{r} \in \mathbb{Z}_n \,:\, \gcd(r,n) = 1\}$ be the group under multiplication modulo n. Then, which one of the following statements is TRUE?

(A) $U(5)$ is isomorphic to $U(8)$

(B) $U(10)$ is isomorphic to $U(12)$

(C) $U(8)$ is isomorphic to $U(10)$

(D) $U(8)$ is isomorphic to $U(12)$

Q.7 Which one of the following is TRUE for the symmetric group S_{13}?

(A) S_{13} has an element of order 42

(B) S_{13} has no element of order 35

(C) S_{13} has an element of order 27

(D) S_{13} has no element of order 60

Q.8 Let G be a finite group containing a non-identity element which is conjugate to its inverse. Then, which one of the following is TRUE?

(A) The order of G is necessarily even

(B) The order of G is not necessarily even

(C) G is necessarily cyclic

(D) G is necessarily abelian but need not be cyclic

Q.9 Consider the following statements.

P: If a system of linear equations $Ax = b$ has a unique solution, where A is an $m \times n$ matrix and b is an $m \times 1$ matrix, then $m = n$.

Q: For a subspace W of a nonzero vector space V, whenever $u \in V \setminus W$ and $v \in V \setminus W$, then $u + v \in V \setminus W$.

Which one of the following holds?

(A) Both P and Q are true

(B) P is true but Q is false

(C) P is false but Q is true

(D) Both P and Q are false

Q.10 Let $g: \mathbb{R} \to \mathbb{R}$ be a continuous function. Which one of the following is the solution of the differential equation

$$\frac{d^2y}{dx^2} + y = g(x) \quad \text{for} \quad x \in \mathbb{R},$$

satisfying the conditions $y(0) = 0,\ y'(0) = 1$?

(A) $y(x) = \sin x - \int_0^x \sin(x-t)\, g(t) dt$

(B) $y(x) = \sin x + \int_0^x \sin(x-t)\, g(t) dt$

(C) $y(x) = \sin x - \int_0^x \cos(x-t)\, g(t) dt$

(D) $y(x) = \sin x + \int_0^x \cos(x-t)\, g(t) dt$

Section A: Q.11 – Q.30 Carry TWO marks each.

Q.11 Which one of the following groups has elements of order $1, 2, 3, 4, 5$ but does not have an element of order greater than or equal to 6 ?

(A) The alternating group A_6

(B) The alternating group A_5

(C) S_6

(D) S_5

Q.12 Consider the group $G = \{A \in M_2(\mathbb{R}) : AA^T = I_2\}$ with respect to matrix multiplication. Let

$$Z(G) = \{A \in G : AB = BA, \text{ for all } \quad B \in G\}.$$

Then, the cardinality of $Z(G)$ is

(A) 1

(B) 2

(C) 4

(D) Infinite

Q.13 Let V be a nonzero subspace of the complex vector space $M_7(\mathbb{C})$ such that every nonzero matrix in V is invertible. Then, the dimension of V over $\mathbb{C}$ is

(A) 1

(B) 2

(C) 7

(D) 49

Q.14 For $n \in \mathbb{N}$, let

$$a_n = \frac{1}{(3n+2)(3n+4)} \quad \text{and} \quad b_n = \frac{n^3 + \cos(3^n)}{3^n + n^3}.$$

Then, which one of the following is TRUE?

(A) $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} b_n$ is divergent

(B) $\sum_{n=1}^{\infty} a_n$ is divergent but $\sum_{n=1}^{\infty} b_n$ is convergent

(C) Both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent

(D) Both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent

Q.15

Let $a = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \\ 0 \end{bmatrix}$. Consider the following two statements.

P: The matrix $I_4 - aa^T$ is invertible.

Q: The matrix $I_4 - 2aa^T$ is invertible.

Then, which one of the following holds?

(A) P is false but Q is true

(B) P is true but Q is false

(C) Both P and Q are true

(D) Both P and Q are false

Q.16 Let A be a 6×5 matrix with entries in $\mathbb{R}$ and B be a 5×4 matrix with entries in $\mathbb{R}$. Consider the following two statements.

P: For all such nonzero matrices A and B, there is a nonzero matrix Z such that AZB is the 6×4 zero matrix.

Q: For all such nonzero matrices A and B, there is a nonzero matrix Y such that BYA is the 5×5 zero matrix.

Which one of the following holds?

(A) Both P and Q are true

(B) P is true but Q is false

(C) P is false but Q is true

(D) Both P and Q are false

Q.17 Let $P_{11}(x)$ be the real vector space of polynomials, in the variable x with real coefficients and having degree at most 11, together with the zero polynomial. Let

$$E = \{s_0(x), s_1(x), \dots, s_{11}(x)\}, \qquad F = \{r_0(x), r_1(x), \dots, r_{11}(x)\}$$

be subsets of $P_{11}(x)$ having 12 elements each and satisfying

$$s_0(3) = s_1(3) = \cdots = s_{11}(3) = 0\,, \qquad r_0(4) = r_1(4) = \cdots = r_{11}(4) = 1\,.$$

Then, which one of the following is TRUE?

(A) Any such E is not necessarily linearly dependent and any such F is not necessarily linearly dependent

(B) Any such E is necessarily linearly dependent but any such F is not necessarily linearly dependent

(C) Any such E is not necessarily linearly dependent but any such F is necessarily linearly dependent

(D) Any such E is necessarily linearly dependent and any such F is necessarily linearly dependent

Q.18 For the differential equation

$$y(8x - 9y)dx + 2x(x - 3y)dy = 0\,,$$

which one of the following statements is TRUE?

(A) The differential equation is not exact and has x^2 as an integrating factor

(B) The differential equation is exact and homogeneous

(C) The differential equation is not exact and does not have x^2 as an integrating factor

(D) The differential equation is not homogeneous and has x^2 as an integrating factor

Q.19 For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x.

For $x, y \in \mathbb{R}$, define

$$\min\{x, y\} = \begin{cases} x & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}$$

Let $f: [-2\pi, 2\pi] \to \mathbb{R}$ be defined by

$$f(x) = \sin(\min\{x, x - \lfloor x \rfloor\}) \text{ for } x \in [-2\pi, 2\pi].$$

Consider the set $S = \{x \in [-2\pi, 2\pi]: f \text{ is discontinuous at } x\}$.

Which one of the following statements is TRUE?

(A) S has 13 elements

(B) S has 7 elements

(C) S is an infinite set

(D) S has 6 elements

Q.20 Define the sequences $\{a_n\}_{n=3}^{\infty}$ and $\{b_n\}_{n=3}^{\infty}$ as

$$a_n = (\log n + \log\log n)^{\log n} \quad \text{and} \quad b_n = n^{\left(1+\frac{1}{\log n}\right)}.$$

Which one of the following is TRUE?

(A) $\sum_{n=3}^{\infty} \frac{1}{a_n}$ is convergent but $\sum_{n=3}^{\infty} \frac{1}{b_n}$ is divergent

(B) $\sum_{n=3}^{\infty} \frac{1}{a_n}$ is divergent but $\sum_{n=3}^{\infty} \frac{1}{b_n}$ is convergent

(C) Both $\sum_{n=3}^{\infty} \frac{1}{a_n}$ and $\sum_{n=3}^{\infty} \frac{1}{b_n}$ are divergent

(D) Both $\sum_{n=3}^{\infty} \frac{1}{a_n}$ and $\sum_{n=3}^{\infty} \frac{1}{b_n}$ are convergent

Q.21 For $p, q, r \in \mathbb{R},\ r \neq 0$ and $n \in \mathbb{N}$, let

$$a_n = p^n\, n^q \left(\frac{n}{n+2}\right)^{n^2} \quad \text{and} \quad b_n = \frac{n^n}{n!\ r^n}\left(\sqrt{\frac{n+2}{n}}\right).$$

Then, which one of the following statements is TRUE?

(A) If $1 < p < e^2$ and $q > 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent

(B) If $e^2 < p < e^4$ and $q > 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent

(C) If $1 < r < e$, then $\sum_{n=1}^{\infty} b_n$ is convergent

(D) If $\frac{1}{e} < r < 1$, then $\sum_{n=1}^{\infty} b_n$ is convergent

Q.22 Let $P_7(x)$ be the real vector space of polynomials, in the variable x with real coefficients and having degree at most 7, together with the zero polynomial. Let $T\colon\ P_7(x) \to P_7(x)$ be the linear transformation defined by

$$T\big(f(x)\big) = f(x) + \frac{df(x)}{dx}\ .$$

Then, which one of the following is TRUE?

(A) T is not a surjective linear transformation

(B) There exists $k \in \mathbb{N}$ such that T^k is the zero linear transformation

(C) 1 and 2 are the eigenvalues of T

(D) There exists $r \in \mathbb{N}$ such that $(T - I)^r$ is the zero linear transformation, where I is the identity map on $P_7(x)$

Q.23 For $\alpha \in \mathbb{R}$, let $y_\alpha(x)$ be the solution of the differential equation

$$\frac{dy}{dx} + 2y = \frac{1}{1+x^2} \quad \text{for } x \in \mathbb{R}$$

satisfying $y(0) = \alpha$. Then, which one of the following is TRUE?

(A) $\lim_{x\to\infty} y_\alpha(x) = 0$ for every $\alpha \in \mathbb{R}$

(B) $\lim_{x\to\infty} y_\alpha(x) = 1$ for every $\alpha \in \mathbb{R}$

(C) There exists an $\alpha \in \mathbb{R}$ such that $\lim_{x\to\infty} y_\alpha(x)$ exists but its value is different from 0 and 1

(D) There is an $\alpha \in \mathbb{R}$ for which $\lim_{x\to\infty} y_\alpha(x)$ does not exist

Q.24 Consider the following two statements.

P: There exist functions $f: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$ such that f is continuous at $x = 1$ and g is discontinuous at $x = 1$ but $g \circ f$ is continuous at $x = 1$.

Q: There exist functions $f: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$ such that both f and g are discontinuous at $x = 1$ but $g \circ f$ is continuous at $x = 1$.

Which one of the following holds?

(A) Both P and Q are true

(B) Both P and Q are false

(C) P is true but Q is false

(D) P is false but Q is true

Q.25 Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \frac{(x^2+1)^2}{x^4+x^2+1} \quad \text{for } x \in \mathbb{R}.$$

Then, which one of the following is TRUE?

(A) f has exactly two points of local maxima and exactly three points of local minima

(B) f has exactly three points of local maxima and exactly two points of local minima

(C) f has exactly one point of local maximum and exactly two points of local minima

(D) f has exactly two points of local maxima and exactly one point of local minimum

Q.26 Let $f: \mathbb{R} \to \mathbb{R}$ be a solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\,\frac{dy}{dx} + y = 2e^x \ \text{ for } \ x \in \mathbb{R}\,.$$

Consider the following statements.

P: If $f(x) > 0$ for all $x \in \mathbb{R}$, then $f'(x) > 0$ for all $x \in \mathbb{R}$.

Q: If $f'(x) > 0$ for all $x \in \mathbb{R}$, then $f(x) > 0$ for all $x \in \mathbb{R}$.

Then, which one of the following holds?

(A) P is true but Q is false

(B) P is false but Q is true

(C) Both P and Q are true

(D) Both P and Q are false

Q.27 For $a > b > 0$, consider

$$D = \left\{(x, y, z) \in \mathbb{R}^3 :\ x^2 + y^2 + z^2 \leq a^2 \ \text{ and } \ x^2 + y^2 \geq b^2\right\}.$$

Then, the surface area of the boundary of the solid D is

(A) $4\pi(a+b)\sqrt{a^2 - b^2}$

(B) $4\pi\left(a^2 - b\sqrt{a^2 - b^2}\right)$

(C) $4\pi(a-b)\sqrt{a^2 - b^2}$

(D) $4\pi\left(a^2 + b\sqrt{a^2 - b^2}\right)$

Q.28 For $n \geq 3$, let a regular n-sided polygon P_n be circumscribed by a circle of radius R_n and let r_n be the radius of the circle inscribed in P_n. Then

$$\lim_{n \to \infty} \left(\frac{R_n}{r_n}\right)^{n^2}$$

equals

(A) $e^{\left(\pi^2\right)}$

(B) $e^{\left(\frac{\pi^2}{2}\right)}$

(C) $e^{\left(\frac{\pi^2}{3}\right)}$

(D) $e^{\left(2\pi^2\right)}$

Q.29 Let L_1 denote the line $y = 3x + 2$ and L_2 denote the line $y = 4x + 3$. Suppose that $f: \mathbb{R} \to \mathbb{R}$ is a four times continuously differentiable function such that the line L_1 intersects the curve $y = f(x)$ at exactly three distinct points and the line L_2 intersects the curve $y = f(x)$ at exactly four distinct points. Then, which one of the following is TRUE?

(A) $\frac{df}{dx}$ does not attain the value 3 on $\mathbb{R}$

(B) $\frac{d^2f}{dx^2}$ vanishes at most once on $\mathbb{R}$

(C) $\frac{d^3f}{dx^3}$ vanishes at least once on $\mathbb{R}$

(D) $\frac{df}{dx}$ does not attain the value $\frac{7}{2}$ on $\mathbb{R}$

Q.30 Define the function $f: \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = 12xy\, e^{-(2x+3y-2)}\,.$$

If (a,b) is the point of local maximum of f, then $f(a,b)$ equals

(A) 2

(B) 6

(C) 12

(D) 0

Section B: Q.31 – Q.40 Carry TWO marks each.

Q.31 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

Then, which of the following statements is/are always TRUE?

(A) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n^2$ converges absolutely

(B) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n^3$ converges absolutely

(C) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges

(D) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^3$ converges

Q.32 Which of the following statements is/are TRUE?

(A) $\displaystyle\sum_{n=1}^{\infty} n \log\left(1+\frac{1}{n^3}\right)$ is convergent

(B) $\displaystyle\sum_{n=1}^{\infty} \left(1-\cos\left(\frac{1}{n}\right)\right)\log n$ is convergent

(C) $\displaystyle\sum_{n=1}^{\infty} n^2 \log\left(1+\frac{1}{n^3}\right)$ is convergent

(D) $\displaystyle\sum_{n=1}^{\infty} \left(1-\cos\left(\frac{1}{\sqrt{n}}\right)\right)\log n$ is convergent

Q.33 Which of the following statements is/are TRUE?

(A) The additive group of real numbers is isomorphic to the multiplicative group of positive real numbers

(B) The multiplicative group of nonzero real numbers is isomorphic to the multiplicative group of nonzero complex numbers

(C) The additive group of real numbers is isomorphic to the multiplicative group of nonzero complex numbers

(D) The additive group of real numbers is isomorphic to the additive group of rational numbers

Q.34 Let $f:(1,\infty)\to(0,\infty)$ be a continuous function such that for every $n\in\mathbb{N}$, $f(n)$ is the smallest prime factor of n. Then, which of the following options is/are CORRECT?

(A) $\lim\limits_{x\to\infty} f(x)$ exists

(B) $\lim\limits_{x\to\infty} f(x)$ does not exist

(C) The set of solutions to the equation $f(x)=2024$ is finite

(D) The set of solutions to the equation $f(x)=2024$ is infinite

Q.35 Let

$$S = \{(x, y) \in \mathbb{R}^2 :\ x > 0, y > 0\},$$

and $f: S \to \mathbb{R}$ be given by

$$f(x, y) = 2x^2 + 3y^2 - \log x - \frac{1}{6}\log y.$$

Then, which of the following statements is/are TRUE?

(A) There is a unique point in S at which $f(x, y)$ attains a local maximum

(B) There is a unique point in S at which $f(x, y)$ attains a local minimum

(C) For each point $(x_0, y_0) \in S$, the set $\{(x, y) \in S: f(x, y) = f(x_0, y_0)\}$ is bounded

(D) For each point $(x_0, y_0) \in S$, the set $\{(x, y) \in S: f(x, y) = f(x_0, y_0)\}$ is unbounded

Q.36 The center $Z(G)$ of a group G is defined as

$$Z(G) = \{x \in G : xg = gx \quad \text{for all} \quad g \in G\}.$$

Let $|G|$ denote the order of G. Then, which of the following statements is/are TRUE for any group G?

(A) If G is non-abelian and $Z(G)$ contains more than one element, then the center of the quotient group $G/Z(G)$ contains only one element

(B) If $|G| \geq 2$, then there exists a non-trivial homomorphism from $\mathbb{Z}$ to G

(C) If $|G| \geq 2$ and G is non-abelian, then there exists a non-identity isomorphism from G to itself

(D) If $|G| = p^3$, where p is a prime number, then G is necessarily abelian

Q.37 For a matrix M, let Rowspace(M) denote the linear span of the rows of M and Colspace(M) denote the linear span of the columns of M. Which of the following hold(s) for all $A, B, C \in M_{10}(\mathbb{R})$ satisfying $A = BC$?

(A) Rowspace$(A) \subseteq$ Rowspace(B)

(B) Rowspace$(A) \subseteq$ Rowspace(C)

(C) Colspace$(A) \subseteq$ Colspace(B)

(D) Colspace$(A) \subseteq$ Colspace(C)

Q.38 Define $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ as follows

$$f(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}\ (m!)^2} \text{ and } g(x) = \frac{x}{2} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}\ (m+1)!\ m!} \quad \text{for } x \in \mathbb{R}.$$

Let $x_1, x_2, x_3, x_4 \in \mathbb{R}$ be such that $0 < x_1 < x_2$, $0 < x_3 < x_4$,

$$f(x_1) = f(x_2) = 0, \qquad f(x) \neq 0 \text{ when } x_1 < x < x_2,$$

$$g(x_3) = g(x_4) = 0 \quad \text{and } g(x) \neq 0 \text{ when } x_3 < x < x_4.$$

Then, which of the following statements is/are TRUE?

(A) The function f does not vanish anywhere in the interval (x_3, x_4)

(B) The function f vanishes exactly once in the interval (x_3, x_4)

(C) The function g does not vanish anywhere in the interval (x_1, x_2)

(D) The function g vanishes exactly once in the interval (x_1, x_2)

Q.39 For $0 < \alpha < 4$, define the sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers as follows:

$$x_1 = \alpha \quad \text{and} \quad x_{n+1} + 2 = -x_n(x_n - 4) \quad \text{for } n \in \mathbb{N}.$$

Which of the following is/are TRUE?

(A) $\{x_n\}_{n=1}^{\infty}$ converges for at least three distinct values of $\alpha \in (0,1)$

(B) $\{x_n\}_{n=1}^{\infty}$ converges for at least three distinct values of $\alpha \in (1,2)$

(C) $\{x_n\}_{n=1}^{\infty}$ converges for at least three distinct values of $\alpha \in (2,3)$

(D) $\{x_n\}_{n=1}^{\infty}$ converges for at least three distinct values of $\alpha \in (3,4)$

Q.40 Consider

$$G = \{m + n\sqrt{2} : m, n \in \mathbb{Z}\}$$

as a subgroup of the additive group $\mathbb{R}$.

Which of the following statements is/are TRUE?

(A) G is a cyclic subgroup of $\mathbb{R}$ under addition

(B) $G \cap I$ is non-empty for every non-empty open interval $I \subseteq \mathbb{R}$

(C) G is a closed subset of $\mathbb{R}$

(D) G is isomorphic to the group $\mathbb{Z} \times \mathbb{Z}$, where the group operation in $\mathbb{Z} \times \mathbb{Z}$ is defined by $(m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2)$

Section C: Q.41 – Q.50 Carry ONE mark each.

Q.41 The area of the region

$$R = \left\{(x,y) \in \mathbb{R}^2 \colon\ 0 \le x \le 1, 0 \le y \le 1 \ \text{ and } \ \frac{1}{4} \le xy \le \frac{1}{2}\right\}$$

is __________ (*rounded off to two decimal places*).

Q.42 Let $y\colon \mathbb{R} \to \mathbb{R}$ be the solution to the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 1$$

satisfying $y(0) = 0$ and $y'(0) = 1$.

Then, $\lim_{x\to\infty} y(x)$ equals __________ (*rounded off to two decimal places*).

Q.43 For $\alpha > 0$, let $y_\alpha(x)$ be the solution to the differential equation

$$2\frac{d^2y}{dx^2} - \frac{dy}{dx} - y = 0$$

satisfying the conditions

$$y(0) = 1, \quad y'(0) = \alpha.$$

Then, the smallest value of α for which $y_\alpha(x)$ has no critical points in $\mathbb{R}$ equals

___________ (*rounded off to the nearest integer*).

Q.44 Consider the 4×4 matrix

$$M = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}.$$

If $a_{i,j}$ denotes the $(i,j)^{\text{th}}$ entry of M^{-1}, then $a_{4,1}$ equals __________ (*rounded off to two decimal places*).

Q.45 Let $P_{12}(x)$ be the real vector space of polynomials in the variable x with real coefficients and having degree at most 12, together with the zero polynomial. Define

$$V = \left\{ f \in P_{12}(x) \colon\ f(-x) = f(x) \text{ for all } x \in \mathbb{R} \ \text{ and } \ f(2024) = 0 \right\}.$$

Then, the dimension of V is ______________

Q.46 Let

$$S = \left\{ f \colon \mathbb{R} \to \mathbb{R} :\ f \text{ is a polynomial and } f\big(f(x)\big) = \big(f(x)\big)^{2024} \ \text{ for } \ x \in \mathbb{R} \right\}.$$

Then, the number of elements in S is ______________

Q.47 Let $a_1 = 1, b_1 = 2$ and $c_1 = 3$. Consider the convergent sequences

$$\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \text{ and } \{c_n\}_{n=1}^{\infty}$$

defined as follows:

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \frac{b_n + c_n}{2} \text{ and } c_{n+1} = \frac{c_n + a_n}{2} \text{ for } n \geq 1.$$

Then,

$$\sum_{n=1}^{\infty} b_n c_n (a_{n+1} - a_n) + \sum_{n=1}^{\infty} (b_{n+1} c_{n+1} - b_n c_n) a_{n+1}$$

equals __________ (*rounded off to two decimal places*)

Q.48 Let

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 4, (x-1)^2 + y^2 \leq 1, z \geq 0\}.$$

Then, the surface area of S equals ___________ (*rounded off to two decimal places*).

Q.49 Let $P_7(x)$ be the real vector space of polynomials in x with degree at most 7, together with the zero polynomial. For $r = 1, 2, \dots, 7$, define

$$s_r(x) = x(x-1)\cdots\big(x-(r-1)\big) \text{ and } s_0(x) = 1.$$

Consider the fact that $B = \{s_0(x), s_1(x), \dots, s_7(x)\}$ is a basis of $P_7(x)$.

If

$$x^5 = \sum_{k=0}^{7} \alpha_{5,k}\, s_k(x)\,,$$

where $\alpha_{5,k} \in \mathbb{R}$, then $\alpha_{5,2}$ equals ___________ (*rounded off to two decimal places*)

Q.50 Let

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 0 & -4 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 2 & 2 \end{pmatrix}.$$

If $p(x)$ is the characteristic polynomial of M, then $p(2) - 1$ equals __________

Section C: Q.51 – Q.60 Carry TWO marks each.

Q.51 For $\alpha \in (-2\pi, 0)$, consider the differential equation

$$x^2 \frac{d^2y}{dx^2} + \alpha x \frac{dy}{dx} + y = 0 \quad \text{for} \quad x > 0.$$

Let D be the set of all $\alpha \in (-2\pi, 0)$ for which all corresponding real solutions to the above differential equation approach zero as $x \to 0^+$. Then, the number of elements in $D \cap \mathbb{Z}$ equals ___________

Q.52 The value of

$$\lim_{t\to\infty}\left(\left(\log\left(t^2 + \frac{1}{t^2}\right)\right)^{-1} \int_1^{\pi t} \frac{\sin^2 5x}{x}\, dx\right)$$

equals ____________ (*rounded off to two decimal places*).

Q.53 Let T be the planar region enclosed by the square with vertices at the points $(0,1), (1,0), (0,-1)$ and $(-1,0)$. Then, the value of

$$\iint_T \Big(\cos(\pi(x-y)) - \cos(\pi(x+y))\Big)^2 \, dx\, dy$$

equals ____________ (*rounded off to two decimal places*).

Q.54 Let

$$S = \{(x,y,z) \in \mathbb{R}^3 \colon\ x^2+y^2+z^2 < 1\}.$$

Then, the value of

$$\frac{1}{\pi}\iiint_S \left((x-2y+z)^2 + (2x-y-z)^2 + (x-y+2z)^2\right) dxdydz$$

equals ________ (*rounded off to two decimal places*).

Q.55 For $n \in \mathbb{N}$, if

$$a_n = \frac{1}{n^3+1} + \frac{2^2}{n^3+2} + \cdots + \frac{n^2}{n^3+n}$$

then the sequence $\{a_n\}_{n=1}^{\infty}$ converges to ____________ (*rounded off to two decimal places*)

Q.56 Consider the function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 - 4x^2 + 4x - 6$.

For $c \in \mathbb{R}$, let

$$S(c) = \left\{ x \in \mathbb{R} : f(x) = c \right\}$$

and $|S(c)|$ denote the number of elements in $S(c)$. Then, the value of

$$|S(-7)| + |S(-5)| + |S(3)|$$

equals _________

Q.57 Let $c > 0$ be such that

$$\int_0^c e^{s^2}\, ds = 3$$

Then, the value of

$$\int_0^c \left(\int_x^c e^{x^2+y^2}\, dy \right) dx$$

equals __________(*rounded off to one decimal place*).

Q.58 For $k \in \mathbb{N}$, let $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1$. A function $f: [0,1] \to \mathbb{R}$ is said to be piecewise linear with nodes $t_1, \ldots, t_k$, if for each $j = 1, 2, \ldots, k+1$, there exist $a_j \in \mathbb{R}$, $b_j \in \mathbb{R}$ such that

$$f(t) = a_j + b_j t \ \text{ for } \ t_{j-1} < t < t_j \,.$$

Let V be the real vector space of all real valued continuous piecewise linear functions on $[0,1]$ with nodes $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$. Then, the dimension of V equals

____

Q.59 For $n \in \mathbb{N}$, let

$$a_n = \frac{1}{n^{n-1}} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} \frac{n^k}{k+1}$$

and $\beta = \lim_{n\to\infty} a_n$. Then, the value of $\log \beta$ equals ___________ (*rounded off to two decimal places*).

Q.60 Define the function $f: (-1,1) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ by

$$f(x) = \sin^{-1} x$$

Let a_6 denote the coefficient of x^6 in the Taylor series of $\left(f(x)\right)^2$ about $x = 0$. Then, the value of $9a_6$ equals ____________ (*rounded off to two decimal places*).